# AP® Calculus AB
# 2012 Scoring Guidelines

The College Board

# AP® CALCULUS AB
# 2012 SCORING GUIDELINES

## Question 1

| $t$ (minutes) | 0 | 4 | 9 | 15 | 20 |
|---|---|---|---|---|---|
| $W(t)$ (degrees Fahrenheit) | 55.0 | 57.1 | 61.8 | 67.9 | 71.0 |

The temperature of water in a tub at time $t$ is modeled by a strictly increasing, twice-differentiable function $W$, where $W(t)$ is measured in degrees Fahrenheit and $t$ is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times $t$ for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

(d) For $20 \le t \le 25$, the function $W$ that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

---

(a) $W'(12) \approx \dfrac{W(15) - W(9)}{15 - 9} = \dfrac{67.9 - 61.8}{6}$

$= 1.017$ (or 1.016)

The water temperature is increasing at a rate of approximately 1.017 °F per minute at time $t = 12$ minutes.

2 : { 1 : estimate; 1 : interpretation with units }

(b) $\int_0^{20} W'(t)\,dt = W(20) - W(0) = 71.0 - 55.0 = 16$

The water has warmed by 16 °F over the interval from $t = 0$ to $t = 20$ minutes.

2 : { 1 : value; 1 : interpretation with units }

(c) $\dfrac{1}{20}\int_0^{20} W(t)\,dt \approx \dfrac{1}{20}(4 \cdot W(0) + 5 \cdot W(4) + 6 \cdot W(9) + 5 \cdot W(15))$

$= \dfrac{1}{20}(4 \cdot 55.0 + 5 \cdot 57.1 + 6 \cdot 61.8 + 5 \cdot 67.9)$

$= \dfrac{1}{20} \cdot 1215.8 = 60.79$

This approximation is an underestimate, because a left Riemann sum is used and the function $W$ is strictly increasing.

3 : { 1 : left Riemann sum; 1 : approximation; 1 : underestimate with reason }

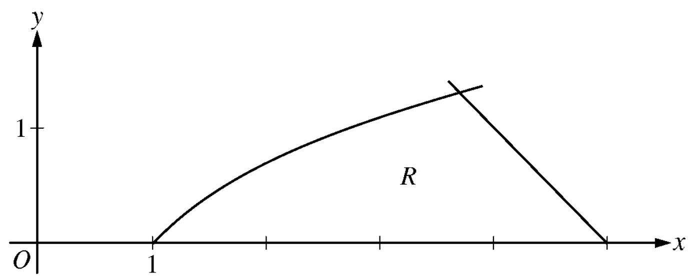
(d) $W(25) = 71.0 + \int_{20}^{25} W'(t)\,dt$

$= 71.0 + 2.043155 = 73.043$

2 : { 1 : integral; 1 : answer }

## Question 2

Let $R$ be the region in the first quadrant bounded by the $x$-axis and the graphs of $y = \ln x$ and $y = 5 - x$, as shown in the figure above.




(a) Find the area of $R$.

(b) Region $R$ is the base of a solid. For the solid, each cross section perpendicular to the $x$-axis is a square. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

(c) The horizontal line $y = k$ divides $R$ into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of $k$.

---

$\ln x = 5 - x \;\Rightarrow\; x = 3.69344$

Therefore, the graphs of $y = \ln x$ and $y = 5 - x$ intersect in the first quadrant at the point $(A, B) = (3.69344, 1.30656)$.

(a) $\text{Area} = \int_0^B \left(5 - y - e^y\right) dy$
$= 2.986 \text{ (or } 2.985)$

OR

$\text{Area} = \int_1^A \ln x \, dx + \int_A^5 (5 - x) \, dx$
$= 2.986 \text{ (or } 2.985)$

3 : { 1 : integrand; 1 : limits; 1 : answer }

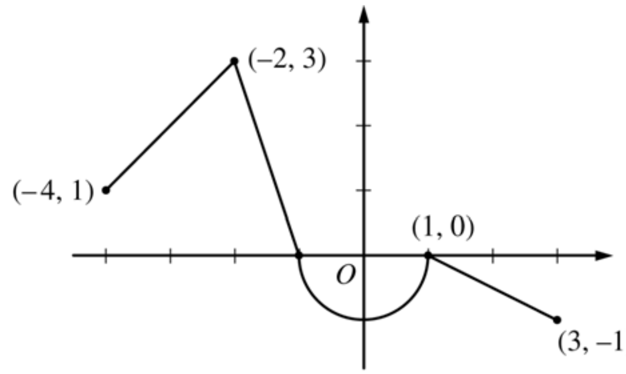
(b) $\text{Volume} = \int_1^A (\ln x)^2 \, dx + \int_A^5 (5 - x)^2 \, dx$

3 : { 2 : integrands; 1 : expression for total volume }

(c) $\int_0^k \left(5 - y - e^y\right) dy = \frac{1}{2} \cdot 2.986 \left(\text{or } \frac{1}{2} \cdot 2.985\right)$

3 : { 1 : integrand; 1 : limits; 1 : equation }

## Question 3




Graph of $f$

Let $f$ be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let $g$ be the function given by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the $x$-coordinate of each point at which the graph of $g$ has a horizontal tangent line. For each of these points, determine whether $g$ has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of $x$ for which the graph of $g$ has a point of inflection. Explain your reasoning.

---

| | |
|---|---|
| (a) $g(2) = \int_1^2 f(t)\,dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$<br>$g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt$<br>$= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$ | $2 : \begin{cases} 1 : g(2) \\ 1 : g(-2) \end{cases}$ |
| (b) $g'(x) = f(x) \;\Rightarrow\; g'(-3) = f(-3) = 2$<br>$g''(x) = f'(x) \;\Rightarrow\; g''(-3) = f'(-3) = 1$ | $2 : \begin{cases} 1 : g'(-3) \\ 1 : g''(-3) \end{cases}$ |
| (c) The graph of $g$ has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$.<br><br>$g'(x)$ changes sign from positive to negative at $x = -1$. Therefore, $g$ has a relative maximum at $x = -1$.<br><br>$g'(x)$ does not change sign at $x = 1$. Therefore, $g$ has neither a relative maximum nor a relative minimum at $x = 1$. | $3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : x = -1 \text{ and } x = 1 \\ 1 : \text{answers with justifications} \end{cases}$ |
| (d) The graph of $g$ has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values. | $2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$ |

# AP® CALCULUS AB
# 2012 SCORING GUIDELINES

## Question 4

The function $f$ is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$.

(b) Write an equation for the line tangent to the graph of $f$ at $x = -3$.

(c) Let $g$ be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x + 7 & \text{for } -3 < x \le 5. \end{cases}$

Is $g$ continuous at $x = -3$ ? Use the definition of continuity to explain your answer.

(d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\, dx$.

---

(a) $f'(x) = \frac{1}{2}\left(25 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{25 - x^2}}, \quad -5 < x < 5$

2 : $f'(x)$

(b) $f'(-3) = \frac{3}{\sqrt{25 - 9}} = \frac{3}{4}$

$f(-3) = \sqrt{25 - 9} = 4$

An equation for the tangent line is $y = 4 + \frac{3}{4}(x + 3)$.

2 : $\begin{cases} 1 : f'(-3) \\ 1 : \text{answer} \end{cases}$

(c) $\lim\limits_{x \to -3^-} g(x) = \lim\limits_{x \to -3^-} f(x) = \lim\limits_{x \to -3^-} \sqrt{25 - x^2} = 4$

$\lim\limits_{x \to -3^+} g(x) = \lim\limits_{x \to -3^+} (x + 7) = 4$

Therefore, $\lim\limits_{x \to -3} g(x) = 4$.

$g(-3) = f(-3) = 4$

So, $\lim\limits_{x \to -3} g(x) = g(-3)$.

Therefore, $g$ is continuous at $x = -3$.

2 : $\begin{cases} 1 : \text{considers one-sided limits} \\ 1 : \text{answer with explanation} \end{cases}$

(d) Let $u = 25 - x^2 \;\Rightarrow\; du = -2x\,dx$

$$\int_0^5 x\sqrt{25 - x^2}\, dx = -\frac{1}{2}\int_{25}^{0} \sqrt{u}\, du$$

$$= \left[-\frac{1}{2} \cdot \frac{2}{3} u^{3/2}\right]_{u=25}^{u=0}$$

$$= -\frac{1}{3}(0 - 125) = \frac{125}{3}$$

3 : $\begin{cases} 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

# AP® CALCULUS AB
# 2012 SCORING GUIDELINES

## Question 5

The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time $t$ days after it is first weighed, then

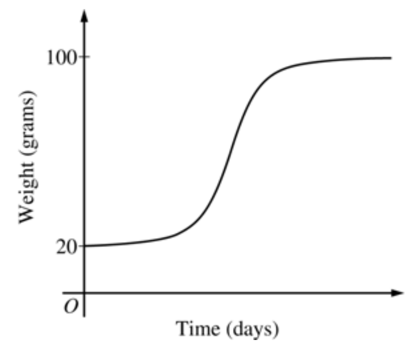
$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of $B$. Use $\frac{d^2B}{dt^2}$ to explain why the graph of $B$ cannot resemble the following graph.

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.




---

(a) $\left.\frac{dB}{dt}\right|_{B=40} = \frac{1}{5}(60) = 12$

$\left.\frac{dB}{dt}\right|_{B=70} = \frac{1}{5}(30) = 6$

Because $\left.\frac{dB}{dt}\right|_{B=40} > \left.\frac{dB}{dt}\right|_{B=70}$, the bird is gaining weight faster when it weighs 40 grams.

$2 : \begin{cases} 1 : \text{uses } \frac{dB}{dt} \\ 1 : \text{answer with reason} \end{cases}$

(b) $\frac{d^2B}{dt^2} = -\frac{1}{5}\frac{dB}{dt} = -\frac{1}{5}\cdot\frac{1}{5}(100 - B) = -\frac{1}{25}(100 - B)$

Therefore, the graph of $B$ is concave down for $20 \le B < 100$. A portion of the given graph is concave up.

$2 : \begin{cases} 1 : \frac{d^2B}{dt^2} \text{ in terms of } B \\ 1 : \text{explanation} \end{cases}$

(c) $\frac{dB}{dt} = \frac{1}{5}(100 - B)$

$\int \frac{1}{100 - B}\, dB = \int \frac{1}{5}\, dt$

$-\ln|100 - B| = \frac{1}{5}t + C$

Because $20 \le B < 100$, $|100 - B| = 100 - B$.

$-\ln(100 - 20) = \frac{1}{5}(0) + C \Rightarrow -\ln(80) = C$

$100 - B = 80e^{-t/5}$

$B(t) = 100 - 80e^{-t/5}, \quad t \ge 0$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } B \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

## Question 6

For $0 \le t \le 12$, a particle moves along the $x$-axis. The velocity of the particle at time $t$ is given by $v(t) = \cos\left(\frac{\pi}{6}t\right)$. The particle is at position $x = -2$ at time $t = 0$.

(a) For $0 \le t \le 12$, when is the particle moving to the left?

(b) Write, but do not evaluate, an integral expression that gives the total distance traveled by the particle from time $t = 0$ to time $t = 6$.

(c) Find the acceleration of the particle at time $t$. Is the speed of the particle increasing, decreasing, or neither at time $t = 4$? Explain your reasoning.

(d) Find the position of the particle at time $t = 4$.

---

| | |
|---|---|
| (a) $v(t) = \cos\left(\frac{\pi}{6}t\right) = 0 \Rightarrow t = 3,\ 9$<br><br>The particle is moving to the left when $v(t) < 0$.<br>This occurs when $3 < t < 9$. | $2 : \begin{cases} 1 : \text{considers } v(t) = 0 \\ 1 : \text{interval} \end{cases}$ |
| (b) $\int_0^6 \lvert v(t) \rvert \, dt$ | 1 : answer |
| (c) $a(t) = -\frac{\pi}{6}\sin\left(\frac{\pi}{6}t\right)$<br><br>$a(4) = -\frac{\pi}{6}\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}\pi}{12} < 0$<br><br>$v(4) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} < 0$<br><br>The speed is increasing at time $t = 4$, because velocity and acceleration have the same sign. | $3 : \begin{cases} 1 : a(t) \\ 2 : \text{conclusion with reason} \end{cases}$ |
| (d) $x(4) = -2 + \int_0^4 \cos\left(\frac{\pi}{6}t\right) dt$<br>$= -2 + \left[\frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right)\right]_0^4$<br>$= -2 + \frac{6}{\pi}\left[\sin\left(\frac{2\pi}{3}\right) - 0\right]$<br>$= -2 + \frac{6}{\pi} \cdot \frac{\sqrt{3}}{2} = -2 + \frac{3\sqrt{3}}{\pi}$ | $3 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$ |