CollegeBoard AP

# AP® Calculus AB
# 2011 Scoring Guidelines

## Question 1

For $0 \le t \le 6$, a particle is moving along the $x$-axis. The particle's position, $x(t)$, is not explicitly given. The velocity of the particle is given by $v(t) = 2\sin\left(e^{t/4}\right) + 1$. The acceleration of the particle is given by $a(t) = \frac{1}{2}e^{t/4}\cos\left(e^{t/4}\right)$ and $x(0) = 2$.

(a) Is the speed of the particle increasing or decreasing at time $t = 5.5$ ? Give a reason for your answer.

(b) Find the average velocity of the particle for the time period $0 \le t \le 6$.

(c) Find the total distance traveled by the particle from time $t = 0$ to $t = 6$.

(d) For $0 \le t \le 6$, the particle changes direction exactly once. Find the position of the particle at that time.

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|---|---|
| (a) $v(5.5) = -0.45337$, $a(5.5) = -1.35851$<br><br>The speed is increasing at time $t = 5.5$, because velocity and acceleration have the same sign. | 2 : conclusion with reason |
| (b) Average velocity $= \frac{1}{6}\int_0^6 v(t)\,dt = 1.949$ | $2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$ |
| (c) Distance $= \int_0^6 \lvert v(t) \rvert\,dt = 12.573$ | $2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$ |
| (d) $v(t) = 0$ when $t = 5.19552$. Let $b = 5.19552$.<br>$v(t)$ changes sign from positive to negative at time $t = b$.<br>$x(b) = 2 + \int_0^b v(t)\,dt = 14.134$ or $14.135$ | $3 : \begin{cases} 1 : \text{considers } v(t) = 0 \\ 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$ |

## Question 2

| $t$ (minutes) | 0 | 2 | 5 | 9 | 10 |
|---|---|---|---|---|---|
| $H(t)$ (degrees Celsius) | 66 | 60 | 52 | 44 | 43 |

As a pot of tea cools, the temperature of the tea is modeled by a differentiable function $H$ for $0 \le t \le 10$, where time $t$ is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time $t$ are shown in the table above.

(a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

(b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

(c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

(d) At time $t = 0$, biscuits with temperature 100°C were removed from an oven. The temperature of the biscuits at time $t$ is modeled by a differentiable function $B$ for which it is known that $B'(t) = -13.84e^{-0.173t}$. Using the given models, at time $t = 10$, how much cooler are the biscuits than the tea?

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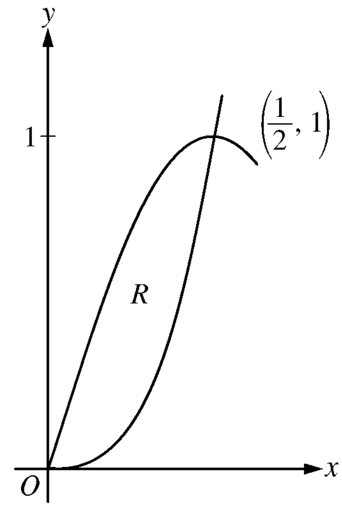
| (a) $H'(3.5) \approx \frac{H(5) - H(2)}{5 - 2} = \frac{52 - 60}{3} = -2.666$ or $-2.667$ degrees Celsius per minute | 1 : answer |
| (b) $\frac{1}{10}\int_0^{10} H(t)\,dt$ is the average temperature of the tea, in degrees Celsius, over the 10 minutes.<br>$\frac{1}{10}\int_0^{10} H(t)\,dt \approx \frac{1}{10}\left(2 \cdot \frac{66+60}{2} + 3 \cdot \frac{60+52}{2} + 4 \cdot \frac{52+44}{2} + 1 \cdot \frac{44+43}{2}\right) = 52.95$ | $3 : \begin{cases} 1 : \text{meaning of expression} \\ 1 : \text{trapezoidal sum} \\ 1 : \text{estimate} \end{cases}$ |
| (c) $\int_0^{10} H'(t)\,dt = H(10) - H(0) = 43 - 66 = -23$<br>The temperature of the tea drops 23 degrees Celsius from time $t = 0$ to time $t = 10$ minutes. | $2 : \begin{cases} 1 : \text{value of integral} \\ 1 : \text{meaning of expression} \end{cases}$ |
| (d) $B(10) = 100 + \int_0^{10} B'(t)\,dt = 34.18275;\quad H(10) - B(10) = 8.817$<br>The biscuits are 8.817 degrees Celsius cooler than the tea. | $3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{uses } B(0) = 100 \\ 1 : \text{answer} \end{cases}$ |

## Question 3




Let $R$ be the region in the first quadrant enclosed by the graphs of $f(x) = 8x^3$ and $g(x) = \sin(\pi x)$, as shown in the figure above.

(a) Write an equation for the line tangent to the graph of $f$ at $x = \frac{1}{2}$.

(b) Find the area of $R$.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when $R$ is rotated about the horizontal line $y = 1$.

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(a) $f\left(\frac{1}{2}\right) = 1$

$f'(x) = 24x^2$, so $f'\left(\frac{1}{2}\right) = 6$

An equation for the tangent line is $y = 1 + 6\left(x - \frac{1}{2}\right)$.

$2 : \begin{cases} 1 : f'\left(\frac{1}{2}\right) \\ 1 : \text{answer} \end{cases}$

(b) 
$$\begin{aligned} \text{Area} &= \int_0^{1/2} (g(x) - f(x))\,dx \\ &= \int_0^{1/2} \left(\sin(\pi x) - 8x^3\right) dx \\ &= \left[-\frac{1}{\pi}\cos(\pi x) - 2x^4\right]_{x=0}^{x=1/2} \\ &= -\frac{1}{8} + \frac{1}{\pi} \end{aligned}$$

$4 : \begin{cases} 1 : \text{integrand} \\ 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) 
$$\begin{aligned} &\pi\int_0^{1/2} \left((1 - f(x))^2 - (1 - g(x))^2\right) dx \\ &= \pi\int_0^{1/2} \left(\left(1 - 8x^3\right)^2 - (1 - \sin(\pi x))^2\right) dx \end{aligned}$$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \end{cases}$

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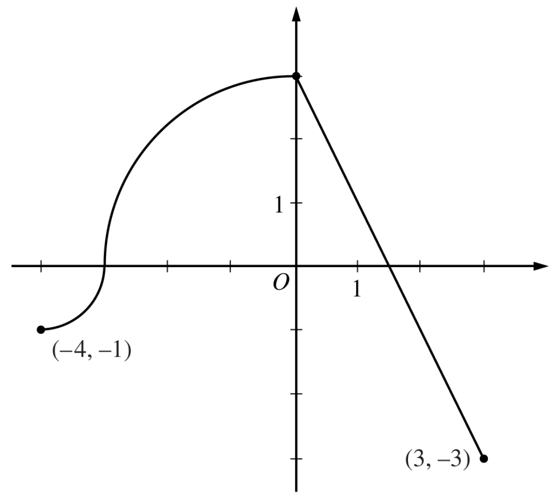
## Question 4

The continuous function $f$ is defined on the interval $-4 \le x \le 3$. The graph of $f$ consists of two quarter circles and one line segment, as shown in the figure above.



Graph of $f$

Let $g(x) = 2x + \int_0^x f(t)\,dt.$

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the $x$-coordinate of the point at which $g$ has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of $x$ on the interval $-4 < x < 3$ for which the graph of $g$ has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of $f$ on the interval $-4 \le x \le 3$. There is no point $c$, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

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| (a) $g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt = -6 - \frac{9\pi}{4}$<br>$g'(x) = 2 + f(x)$<br>$g'(-3) = 2 + f(-3) = 2$ | $3 : \begin{cases} 1 : g(-3) \\ 1 : g'(x) \\ 1 : g'(-3) \end{cases}$ |
| (b) $g'(x) = 0$ when $f(x) = -2$. This occurs at $x = \frac{5}{2}$.<br>$g'(x) > 0$ for $-4 < x < \frac{5}{2}$ and $g'(x) < 0$ for $\frac{5}{2} < x < 3$.<br>Therefore $g$ has an absolute maximum at $x = \frac{5}{2}$. | $3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : \text{identifies interior candidate} \\ 1 : \text{answer with justification} \end{cases}$ |
| (c) $g''(x) = f'(x)$ changes sign only at $x = 0$. Thus the graph of $g$ has a point of inflection at $x = 0$. | 1 : answer with reason |
| (d) The average rate of change of $f$ on the interval $-4 \le x \le 3$ is $\frac{f(3) - f(-4)}{3 - (-4)} = -\frac{2}{7}$.<br>To apply the Mean Value Theorem, $f$ must be differentiable at each point in the interval $-4 < x < 3$. However, $f$ is not differentiable at $x = -3$ and $x = 0$. | $2 : \begin{cases} 1 : \text{average rate of change} \\ 1 : \text{explanation} \end{cases}$ |

## Question 5

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function $W$ models the total amount of solid waste stored at the landfill. Planners estimate that $W$ will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. $W$ is measured in tons, and $t$ is measured in years from the start of 2010.

(a) Use the line tangent to the graph of $W$ at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of $W$. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

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(a) $\left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$

The tangent line is $y = 1400 + 44t$.

$W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411$ tons

$2 : \begin{cases} 1 : \frac{dW}{dt} \text{ at } t = 0 \\ 1 : \text{answer} \end{cases}$

(b) $\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \geq 1400$

Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \leq t \leq \frac{1}{4}$.

The answer in part (a) is an underestimate.

$2 : \begin{cases} 1 : \frac{d^2W}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$

(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

$\int \frac{1}{W - 300}\, dW = \int \frac{1}{25}\, dt$

$\ln|W - 300| = \frac{1}{25}t + C$

$\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$

$W - 300 = 1100e^{\frac{1}{25}t}$

$W(t) = 300 + 1100e^{\frac{1}{25}t}, \quad 0 \leq t \leq 20$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } W \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

## Question 6

Let $f$ be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

(a) Show that $f$ is continuous at $x = 0$.

(b) For $x \ne 0$, express $f'(x)$ as a piecewise-defined function. Find the value of $x$ for which $f'(x) = -3$.

(c) Find the average value of $f$ on the interval $[-1, 1]$.

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(a) $\lim\limits_{x\to 0^-} (1 - 2\sin x) = 1$

$\lim\limits_{x\to 0^+} e^{-4x} = 1$

$f(0) = 1$

So, $\lim\limits_{x\to 0} f(x) = f(0)$.

Therefore $f$ is continuous at $x = 0$.

2 : analysis

(b) $f'(x) = \begin{cases} -2\cos x & \text{for } x < 0 \\ -4e^{-4x} & \text{for } x > 0 \end{cases}$

$-2\cos x \ne -3$ for all values of $x < 0$.

$-4e^{-4x} = -3$ when $x = -\frac{1}{4}\ln\left(\frac{3}{4}\right) > 0$.

Therefore $f'(x) = -3$ for $x = -\frac{1}{4}\ln\left(\frac{3}{4}\right)$.

$3 : \begin{cases} 2 : f'(x) \\ 1 : \text{value of } x \end{cases}$

(c) $$\begin{aligned} \int_{-1}^{1} f(x)\,dx &= \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx \\ &= \int_{-1}^{0} (1 - 2\sin x)\,dx + \int_{0}^{1} e^{-4x}\,dx \\ &= \Big[x + 2\cos x\Big]_{x=-1}^{x=0} + \left[-\frac{1}{4}e^{-4x}\right]_{x=0}^{x=1} \\ &= (3 - 2\cos(-1)) + \left(-\frac{1}{4}e^{-4} + \frac{1}{4}\right) \end{aligned}$$

$$\begin{aligned} \text{Average value} &= \frac{1}{2}\int_{-1}^{1} f(x)\,dx \\ &= \frac{13}{8} - \cos(-1) - \frac{1}{8}e^{-4} \end{aligned}$$

$4 : \begin{cases} 1 : \int_{-1}^{0} (1 - 2\sin x)\,dx \text{ and } \int_{0}^{1} e^{-4x}\,dx \\ 2 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$